Noether theorem for μ-symmetries

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Abstract

We give a version of Noether theorem adapted to the framework of μ-symmetries; this extends to such case recent work by Muriel, Romero and Olver in the framework of λ-symmetries, and connects μ-symmetries of a Lagrangian to a suitably modified conservation law. In some cases this 'μ-conservation law' actually reduces to a standard one; we also note a relation between μ-symmetries and conditional invariants. We also consider the case where the variational principle is itself formulated as requiring vanishing variation under μ-prolonged variation fields, leading to modified Euler–Lagrange equations. In this setting, μ-symmetries of the Lagrangian correspond to standard conservation laws as in the standard Noether theorem. We finally propose some applications and examples.

PACS numbers: 02.30.Hq, 02.20.Sv

Introduction

The study of nonlinear differential equations was the main motivation to Sophus Lie when he created what is nowadays known as the theory of Lie groups and Lie algebras. After a period of near-oblivion, symmetry methods are now recognized as one of the most effective tools for the study of differential equations—both in geometrical sense and for what concerns the search for explicit solutions [11, 18, 31, 37].

The relevance of symmetry properties in analytical mechanics and field theory underwent a similar parabola; after Emmy Noether established her theorem [30], it has somehow been overlooked for some time, and then came to be recognized as a tremendously important tool, actually at the basis of all the conservation laws of physics (see [17] for an history of the appraisal of Noether theorem, as well as for an in-depth discussion of it). It is thus not surprising at all that Noether theorem is a central issue not only in treatments of analytical mechanics and field theory [6, 19, 20], but also in the treatment of symmetry methods for differential equations, in particular when one is concerned with equations arising from a variational principle [31].

The effectiveness of symmetry methods for differential equations led various authors to consider generalizations in several directions (which we will not discuss in general). Here we are specially interested in one of these, first proposed by M C Muriel and J L Romero under the name of '$C^\infty(M^{(1)})$-symmetries' or 'C^∞-symmetries' for short, or finally of 'λ-symmetries', in the framework of ODEs [24], and then generalized to PDEs under the name of 'μ-symmetries' [10, 12, 14].

It should be stressed that in this case the generalization with respect to standard Lie symmetries lies not in the definition of 'symmetry', but in the way vector fields are prolonged from the phase manifold $\mathcal{M}$ (the manifold of dependent and independent variables, see below) to jet manifolds $J^r\mathcal{M}$ of appropriate order for the equation at hand. In this, they are different from other proposed generalizations of Lie symmetries.

Definition and properties of μ-prolongations and symmetries (which include λ-ones as a special case) are recalled and discussed in section 2, so here we just recall that these allow us to integrate by symmetry methods equations which are known not to admit any standard Lie symmetry [24, 25]; needless to say, this is a major reason (but not the only one) for the interest they raised.

It is then natural to wonder if Noether theorem can be somehow extended[3] to encompass λ- or μ-symmetries as well. This question was tackled by Muriel, Romero and Olver [27], who were able to give an adapted formulation of Noether theorem relating λ-symmetries of the Lagrangian $\mathcal{L}$ defining the variational problem and conservation laws in a suitably modified sense for problems giving raise to ODEs—the latter being of course the Euler–Lagrange equations for $\mathcal{L}$.

In the present paper we have a twofold aim. On the one hand, we extend the results obtained by Muriel, Romero and Olver in the case of λ-symmetries—hence mechanics and ODEs—to the case of μ-symmetries and hence of field theory and PDEs. In this way we show that a μ-symmetry of the Lagrangian leads to a modified 'conservation law'—called a μ-conservation law—under the evolution law described by the standard Euler–Lagrange equations for $\mathcal{L}$. In some cases, the μ-conservation law actually reduces to a standard one. We also note a relation between μ-symmetries and *conditional invariants*.

On the other hand, we also want to raise a different question. That is, the Euler–Lagrange equations are obtained under the assumption that given a variation δu of the dependent variables u, partial derivatives of these are varied accordingly; this corresponds to considering a 'variation field' Δ (in the simplest case, $\Delta = \varphi^a(u)(\partial/\partial u^a)$) acting in the phase manifold $\mathcal{M}$, and lifting it to the first jet space $J^1\mathcal{M}$—or to higher order jet space for Lagrangians of order higher than one—according to the standard prolongation law. The requirement of zero variation for the action integral leads then, in the familiar way, to establishing the Euler–Lagrange equations.

But, as recalled above, in λ- and μ-symmetries one is actually modifying the prolongation operation; it is then rather natural to consider what happens when the requirement of zero variation for the action integral is considered under variation vector fields which are lifted to jet manifolds with the modified prolongation operation. In this case, one obtains a modified version of the Euler–Lagrange equations—which will be christened as the μ-Euler–Lagrange equations—and hence a *modified* evolution law. In the final part of the present paper, we show that when we consider this modified evolution law, a μ-symmetry of the Lagrangian leads to a standard conservation law. This will also be interpreted in the light of the findings of [10].

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[3] The original Noether theorem was actually already much more general than the simple version usually met in mechanics textbooks [6, 19], as discussed at length in [17, 31]. Several extensions of Noether theorem have also been given in the literature, see e.g. [17, 31, 35]. Here we limit our discussion to the simpler forms of the Noether theorem.

We choose to mostly limit our discussion to first-order Lagrangians; this case—besides being of special physical interest—allows a simpler notation and does not hide behind complicate formulae the main point of our discussion, which would extend to higher order Lagrangians in the same way as done for the standard Noether theorem. Similarly, our examples are as simple as possible in order to illustrate the results and the main points of our discussion.

The paper is organized as follows: in section 1, we establish some general notation; in section 2, we recall the basics of μ-prolongation and μ-symmetries, and define μ-conservation laws. In section 3, we establish a version of Noether theorem for μ-symmetries; this yields correspondence with a μ-conservation law. In section 4, we extend this result to the case of divergence μ-symmetries. In section 5, we establish the connection between μ-symmetries and (standard) conditional invariants. Section 6 is devoted to recalling the gauge equivalence between μ-prolongation and standard ones; this allow us to establish that in a number of cases—but not in general!—the μ-conservation law associated with a μ-symmetry of the Lagrangian is actually a standard conservation law. In section 7, we establish the μ-Euler–Lagrange equations for a Lagrangian $\mathcal{L}$, and show that a μ-symmetry of $\mathcal{L}$ yields a standard conservation law under such an evolution law; this result is also seen in relation with the gauge equivalence mentioned above. Section 8 is devoted to several illustrative examples and applications; some short conclusions will be presented in section 9.

1. General notation

We will preliminarily set up some general notation, to be used as a default in the following [11, 18, 31, 37].

We will denote independent variables as $x = (x^1, \ldots, x^p) \in B \subseteq \mathbf{R}^p$; similarly we denote dependent variables (fields) as $u = (u^1, \ldots, u^q) \in U \subseteq \mathbf{R}^q$. The total manifold of dependent and independent variables will be denoted as $\mathcal{M} = B \times U$; it has the structure of a fiber bundle over B, and one will also (implicitly) consider the associated jet bundles $J^k\mathcal{M}$. In the following, we will customarily assume that actually $B = \mathbf{R}^p$, $U = \mathbf{R}^q$ (and hence $\mathcal{M}$ is also a linear space), as this will simplify some notation; the discussion could dispense with this assumption with no extra difficulty but a heavier notation in sections 2.3 and 6.1.

We use indices $i, j, k, \ldots$ for independent variables, and $a, b, c, \ldots$ for dependent ones (thus denoted respectively e.g. by x^i and u^a). Sum over repeated indices (and multi-indices, see below) is always understood. The partial derivative of u^a with respect to x^i will be denoted in shorthand notation as u^a_i. Total derivative operators will be denoted as D_i. We employ whenever appropriate multi-index notation; a generic multi-index will be denoted as $J = (j_1, \ldots, j_p)$, and its order by $|J| := j_1 + \cdots + j_p$. By u^a_J one means $(\partial^{|J|}u^a / (\partial^{j_1} x^1 \cdots \partial^{j_p} x^p))$; by D_J one means $D_1^{j_1} \cdots D_p^{j_p}$.

Vector fields in $\mathcal{M}$ will usually be written (in coordinates) as

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \varphi^a(x, u)\frac{\partial}{\partial u^a}. \tag{1}$$

The rth-order (standard) prolongation [11, 18, 31, 37] of such a vector field will be usually written as (here the sum is over all multi-indices with $|J| \leqslant r$)

$$X^{(r)} = \xi^i(x, u)\frac{\partial}{\partial x^i} + \varphi^a_J(x, u)\frac{\partial}{\partial u^a_J}, \tag{2}$$

the coefficients φ^a_J satisfy $\varphi^a_0 = \varphi^a$ and the standard prolongation formula

$$\varphi^a_{J,k} = D_k \varphi^a_J - u^a_{J,i} D_k \xi^i. \tag{3}$$

We will deal with Lagrangians; albeit these could depend on x^i, we will usually ignore this dependence, and write $\mathcal{L} = \mathcal{L}(u, u_x)$ when we want to stress these are first-order Lagrangians. The momenta are generally denoted by π, with

$$\pi_a^i := \frac{\partial \mathcal{L}}{\partial u_i^a}.$$

Finally, several of the objects we consider will take value in a set of matrices (usually corresponding to the representation of a Lie algebra or group); we denote this by saying they are M-functions. Thus an M-vector will be a vector whose component take value on a space of matrices.

2. μ-prolongations, μ-symmetries and μ-conservation laws

We will first give general definitions for μ-prolongations and μ-symmetries. Later on, in section 6, we briefly recall their relation with gauge transformations.

We refer to [10, 12, 14] for further details on μ-prolongations and μ-symmetries. These include as special cases the λ-prolongations and λ-symmetries; see [24–27, 34] for further detail on these.

2.1. μ-prolongations

In μ-prolongations, one equips $\mathcal{M}$ with a semi-basic form

$$\mu = \Lambda_i(x, u, u_x) \, dx^i \tag{4}$$

with Λ_i a function taking values in a representation $T_{\mathcal{G}}$ of a Lie algebra $\mathcal{G}$ acting in F. (In the following, we will identify the representation and the Lie algebra for ease of language, hence call μ a $\mathcal{G}$-valued form rather than a $T_{\mathcal{G}}$-valued one.)

It should be stressed that the Λ_i ($i = 1, \ldots, p$) are square ($q \times q$) matrices; with the notation introduced above, we also say that they are M-functions of the basic dependent variables u and independent ones x, and of the first derivatives u_x.

For $p = 1$ and $\Lambda = \lambda I$ (with $\lambda(x, u, u_x)$ a scalar function and I the identity matrix) the M-function Λ actually corresponds to a standard scalar function λ, and we recover the setting of λ-symmetries [24]. Note that we get a very similar setting even in the case where $p > 1$, but the Λ_i are of the form $\Lambda_i = \lambda_i I$, with $\lambda_i = \lambda_i(x, u, u_x)$ scalar functions [26].

The $\mathcal{G}$-valued functions Λ_i should satisfy the compatibility condition

$$D_i \Lambda_j - D_j \Lambda_i + [\Lambda_i, \Lambda_j] = 0, \tag{5}$$

where D_i is the total derivative with respect to x_i; equivalently (but in coordinates-free notation), the $\mathcal{G}$-valued form μ must satisfy the *horizontal Maurer–Cartan equation*

$$D\mu + \tfrac{1}{2}[\mu, \mu] = 0. \tag{6}$$

The μ-prolonged vector field acting in $J^{(r)}\mathcal{M}$ will be denoted as $X^{(r)}_{(\mu)}$, i.e.

$$X^{(r)}_{(\mu)} = \xi^i \frac{\partial}{\partial x^i} + \psi_J^a \frac{\partial}{\partial u_J^a} \tag{7}$$

where sum over repeated indices and derivation multi-indices (of order $|J| \leqslant r$) is implied, and where the familiar prolongation formula (3) is now replaced by

$$\psi_{J,k}^a = D_k \psi_J^a - u_{J,i}^a D_k \xi^i + (\Lambda_k)^a{}_b \big(\psi_J^b - u_{J,i}^b \xi^i\big) \tag{8}$$

with $\psi_0^a = \varphi^a$. This is also rewritten as

$$\psi_{J,k}^a = \nabla_k \psi_J^a - u_{J,i}^a \nabla_k \xi^i, \tag{9}$$

where we have introduced the $\mathcal{G}$-valued differential operators

$$\nabla_k := \delta^a{}_b D_k + (\Lambda_k)^a{}_b. \tag{10}$$

The operators ∇_i will play a key role in our constructions. We stress that for $\Lambda_i = 0$ the ∇_i reduce to the familiar total derivative operators D_i; in general, the ∇_i take the place of the D_i when we consider μ-prolongations (and related concepts) instead of standard ones.

Note that the compatibility condition (5) is written, with the notation (10), as the zero-curvature condition

$$[\nabla_i, \nabla_j] = 0. \tag{11}$$

Formulae become simpler when we deal with *evolutionary representatives* of vector fields; these are vertical for the fibration $(\mathcal{M}, \pi, B)$, and the evolutionary representative of X reads

$$X_Q = Q^a(x, u, u_x)\frac{\partial}{\partial u^a} \qquad \text{with} \qquad Q^a := \varphi^a - \xi^i u_i^a. \tag{12}$$

With this, the recursion formula (9) for the μ-prolongation becomes, with obvious notation,

$$Q_J^a = \nabla_J Q^a. \tag{13}$$

2.2. μ-symmetries

Given a system Δ of differential equations of order r, call S_Δ the corresponding solution manifold in $J^r\mathcal{M}$. We say that X is a μ-symmetry of the system Δ (for the given form μ) if $X_{(\mu)}^{(r)} : S_\Delta \to \mathrm{T}S_\Delta$.

We will often use Y as a shorthand notation for $X_{(\mu)}^{(r)}$.

Similarly, given a function $\mathcal{F} : J^r\mathcal{M} \to \mathbf{R}$, we say that X is a μ-symmetry for $\mathcal{F}$ (or equivalently that $\mathcal{F}$ is μ-invariant under X) if $Y[\mathcal{F}] = 0$; this is also expressed as the invariance of level manifolds of $\mathcal{F}$ under Y.

We stress that—as discussed in [10]—μ-symmetries are not symmetries in the usual sense; in particular, they do not in general transform solutions of the differential problem into other solutions. (Moreover, λ- and μ-symmetries of a given equation do not in general form a Lie algebra.) It was shown in [24, 26] and then in [10, 14] that they are essentially as good as standard symmetries as far as determining exact solutions of the differential problem is concerned. We will show, generalizing some previous results [27], that they can also be quite useful in the context of Noether theory.

In the following, we will deal with vector fields X having $\xi^i = 0$, i.e. $Q^a \equiv \varphi^a$; these represent transformations in the space of dependent variables (fields) not affecting nor depending on independent variables (note this is the same class of transformations considered in the standard formulation of the classical Noether theorem [6, 19]). This assumption greatly simplifies the formulae, in particular the expression of the invariance condition of the Lagrangian, and of the conservation laws.

In section 4, we will show how to extend our results to the case of divergence symmetries; as well known, see [31], this can allow us to include in our discussion not only vector fields with $\xi^i \neq 0$ (introducing evolutionary vector fields), but also the case of generalized vector fields.

2.3. Reduction of μ-prolongation to ordinary and λ ones

We can write the coefficients ψ_J^a of the μ-prolongation (8) as

$$\psi_J^a = \varphi_J^a + F_J^a$$

where φ_J^a are the coefficients of the standard prolongation (2,3); inspection of the standard and μ-prolongation formulae in recursive form (see [14] for details) shows that the recursion formula for the μ-difference terms F_J^a is

$$F_{J,i}^a = \left[\delta^a{}_b\, D_i + (\Lambda_i)^a{}_b\right]F_J^b + (\Lambda_i)^a{}_b\, D_J Q^b = \nabla_i F_J^a + (\Lambda_i)^a{}_b\, D_J Q^b$$

with of course $F_0^a = 0$. It follows, as discussed at length in [10], that standard and μ-prolongations coincide on vector fields such that the Q^a belong identically (hence with all their partial derivatives) to the null space of all the matrices Λ_i.

In particular, if the Λ_i belong to a representation $T_{\mathcal{G}}$ of the Lie algebra $\mathcal{G}$, acting in the $M = \mathbf{R}^p \times \mathbf{R}^q$ space by a linear and non-free action via this, and we call $M_0 \subseteq T\mathcal{M}$ the subspace fixed under all of $T_{\mathcal{G}}$, then all vector fields having only components along TM_0 will be in this class.

An interesting 'intermediate' case arises when the Λ_i admit common eigenspaces, in general with different eigenvalues for different Λ_i (see also subsection 6.4 in this respect). In fact, suppose there is a subspace M_1 such that $\Lambda_i \mathbf{v} = \lambda_i \mathbf{v}$ for all $i = 1, \dots, p$ and all $\mathbf{v} \in TM_1$. Then—as long as we consider vector fields such that Q^a belongs identically to TM_1, for all $a = 1, \dots, q$—things behave as if we were in the case $\Lambda_i = \lambda_i I$, i.e. by all means we deal with the case of 'vector' λ-prolongations and symmetries [26].

2.4. μ-conservation laws

A (standard) *conservation law* is a relation

$$D_i \mathbf{P}^i = 0, \tag{14}$$

where $\mathbf{P}^i$ is a p-dimensional vector.

In our case, we define a **μ-conservation law** as a relation

$$\mathrm{Tr}(\nabla_i \mathcal{P}^i) = 0 \tag{15}$$

involving the $\mathcal{G}$-valued differential operators ∇_i and some $\mathcal{G}$-valued M-vector $\mathcal{P}$. In components, (15) reads

$$(\nabla_i)^a{}_b\,(\mathcal{P}^i)^b{}_a \equiv \left(\nabla^a{}_b\right)_i \left(\mathcal{P}^b{}_a\right)^i = 0. \tag{16}$$

Thus $\mathcal{P}^i$ is a (matrix-valued) vector whose divergence with respect to the (matrix) differential operator ∇ vanishes; note here the divergence is defined with the help of the trace operation, see (15). In this case the M-vector $\mathcal{P}$ will be called a μ-*conserved vector*.

Note that μ-conservation does not imply conservation of $\mathcal{P}$ nor of its trace in ordinary sense; if $\mathcal{P}$ is a μ-conserved vector, then—putting now $\mathbf{P}^i = \mathrm{Tr}(\mathcal{P}^i)$ and recalling the definition (10)—it satisfies

$$D_i \mathbf{P}^i \equiv D_i (\mathcal{P}^i)^a{}_a = -(\Lambda_i)^a{}_b\,(\mathcal{P}^i)^b{}_a = -\mathrm{Tr}(\Lambda_i \mathcal{P}^i). \tag{17}$$

This takes the place of (14), and of course reduces to it for $\Lambda_i = 0$, i.e. $\mu = 0$.

3. Noether theorem for μ-symmetries

The relation between standard symmetries of the Lagrangian and conservation laws is described by the classical Noether theorem [17, 30]. Muriel, Romero and Olver recently gave a Noether-type theorem for λ-symmetries. Here we extend it to the case of μ-symmetries. Our first result is that when the exact invariance of the Lagrangian is replaced by a μ-invariance, then the standard Noether theorem is replaced by a suitably 'corrected' form. Using the operator ∇_i, this μ-conservation law can be written in the form (15), formally similar to a standard

conservation law; alternatively, the rhs of (17) can be interpreted as the 'deviation from the standard conservation law' (i.e. from the case $\mu = 0$).

It is convenient to discuss separately the case of first-order Lagrangians.

Theorem 1. *Consider the first-order Lagrangian $\mathcal{L}(u, u_x)$ and the vector field $X = \varphi^a (\partial / \partial u^a)$. Define*

$$\left(\mathcal{P}^a{}_b \right)^i := \varphi^a \pi^i_b, \tag{18}$$

where $\pi^i_a := \left(\partial \mathcal{L} / \partial u^a_i \right)$, then X is a μ-symmetry for $\mathcal{L}$ if and only if the M-vector $\mathcal{P}^i$ is a μ-conserved vector.

Proof. We observe preliminarily that for first-order Lagrangians, the μ-conservation of $\left(\mathcal{P}^a{}_b \right)^i$ under X (with the notation used above) is equivalent to

$$(\nabla_i)^a{}_b \, (\mathcal{P}^i)^b{}_a = D_i \operatorname{Tr}(\mathcal{P}^i) + (\Lambda_i)^a{}_b \, \varphi^b \frac{\partial \mathcal{L}}{\partial u^a_i} = D_i \mathbf{P}^i + \operatorname{Tr}(\Lambda_i \mathcal{P}^i) = 0, \tag{19}$$

this is easily checked by direct computation.

Let us first show that μ-invariance of the Lagrangian implies μ-conservation of $\mathcal{P}^a{}_b$, we recall that this conservation can be expressed in the form (19). As $X = \varphi^a (\partial / \partial u^a)$, its first μ-prolongation is

$$X^{(1)}_{(\mu)} = \varphi^a \frac{\partial}{\partial u^a} + \left[D_i \varphi^a + (\Lambda_i)^a{}_b \, \varphi^b \right] \frac{\partial}{\partial u^a_i}. \tag{20}$$

Applying this on the Lagrangian $\mathcal{L}$ and integrating by parts, we get

$$X^{(1)}_{(\mu)}[\mathcal{L}] = \varphi^a \left(\frac{\partial \mathcal{L}}{\partial u^a} - D_i \frac{\partial \mathcal{L}}{\partial u^a_i} \right) + D_i \left(\varphi^a \pi^i_a \right) + (\Lambda_i)^a{}_b \, \varphi^b \pi^i_a, \tag{21}$$

the Euler–Lagrange equations $\mathcal{E}[\mathcal{L}] = 0$ grant that the first term vanishes on solutions to the equations, hence this reduces to

$$X^{(1)}_{(\mu)}[\mathcal{L}] = D_i \left(\varphi^a \pi^i_a \right) + (\Lambda_i)^a{}_b \, \varphi^b \pi^i_a. \tag{22}$$

Recalling the definition of ∇_i, see (10), we recognize this as

$$X^{(1)}_{(\mu)}[\mathcal{L}] = (\nabla_i)^a{}_b \, \pi^i_a \varphi^b. \tag{23}$$

This shows that $X^{(1)}_{(\mu)}[\mathcal{L}] = 0$ implies (19). Going through this computation the other way round, we also show that μ-conservation of $\mathcal{P}$ implies $X^{(1)}_{(\mu)}[\mathcal{L}] = 0$. $\qquad\square$

The formulation of theorem 1 is limited to first-order Lagrangians. This is often also the case when one quotes the standard Noether theorem; actually, Noether theory—i.e. the relation between symmetries and conservation laws—also holds for higher order Lagrangians, albeit with a necessarily more complex notation[4]. The same holds for our generalization, i.e. theorem 1 extends to higher order Lagrangians.

We state without proof the following result, which can be verified using a suitable recursive procedure (see [27] for the case $p = q = 1$). Note this extends standard Noether theory via a 'minimal substitution'.

Theorem 2. *Consider the rth-order Lagrangian $\mathcal{L} = \mathcal{L}(x, u^{(r)})$, and a vector field X. Then*

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[4] In view of this fact, our subsequent results will be stated for first-order Lagrangians. Extensions along the lines of the extensions of theorem 1 as provided by theorem 2 would also be possible; however, these would not involve new ideas, and would instead involve notational heaviness, so that we avoid to state these.

(a) *X is a μ-symmetry for $\mathcal{L}$, i.e. $Y[\mathcal{L}] = 0$, if and only if there exists M-vector $(\mathcal{P}^i)^b{}_a$ satisfying the μ-conservation law*

$$(\nabla_i)^a{}_b (\mathcal{P}^i)^b{}_a = 0. \tag{24}$$

(b) *The M-vector $(\mathcal{P}^i)^b{}_a$ is obtained in the following way: write the usual current density vector $(\mathbf{P}_0)^i$ as determined by the given Lagrangian and by the vector field X considered as a standard symmetry for $\mathcal{L}$ (i.e. with $\Lambda_i \equiv 0$), and replace each term $(D_J \varphi^a)$ appearing in $(\mathbf{P}_0)^i$ with $(\nabla_J \varphi)^a$ $(|J| \geqslant 0)$;*

(c) *the conservation law (24) is obtained replacing the global divergence operator D_i with $(\nabla_i)^b{}_a$.*

This result holds for Lagrangians of any order. For first-order Lagrangians the explicit form of the μ-conservation law was given above, see equation (19) and theorem 1. For second-order Lagrangians, the μ-conserved vector is

$$\left(\mathcal{P}^a{}_b\right)^i = \varphi^b \frac{\partial \mathcal{L}}{\partial u^a_i} + \left((\nabla_j)^b{}_c \varphi^c\right) \frac{\partial \mathcal{L}}{\partial u^a_{ij}} - \varphi^b D_j \frac{\partial \mathcal{L}}{\partial u^a_{ij}}. \tag{25}$$

It is easy to check directly that the lhs of the equation $\nabla_i \mathcal{P}^i = 0$, in both these cases, i.e. with $\mathcal{P}$ given by (18) and respectively by (25), is in fact equal to

$$Y[\mathcal{L}] + \varphi \cdot \mathcal{E}(\mathcal{L}) \tag{26}$$

where $\mathcal{E}$ is the Euler–Lagrange operator.

For the vector (25) the μ-conservation law (24) can be written as $D_i(\mathbf{P}^i) = -(\Lambda_i)^a{}_b (\mathcal{P}^i)^b{}_a$, as in (17); alternatively, if one introduces the matrix-valued vector $(\mathcal{P}^i_0)^a{}_b$ which is obtained putting $\Lambda_i = 0$ in $(\mathcal{P}^i)^a{}_b$, the μ-conservation law (24) can also be written as

$$D_i\left(\mathbf{P}^i_0\right) = -D_i\left((\Lambda_j \varphi)_a \frac{\partial \mathcal{L}}{\partial u^a_{ij}}\right) - (\Lambda_i)^a{}_b \left(\mathcal{P}^i_0\right)^b{}_a - (\Lambda_i \Lambda_j)^a{}_b \varphi_b \frac{\partial \mathcal{L}}{\partial u^a_{ij}} \tag{27}$$

where the contribution due to the Λ matrices is entirely shifted at the rhs.

4. Divergence μ-symmetries

In the standard Noether theorem, a conserved current can be related, as well known, to a *divergence symmetry* of the Lagrangian, i.e. to a vector field $X = \xi^i(\partial/\partial x^i) + \varphi^a(\partial/\partial u^a)$ such that $X^{(1)}[\mathcal{L}] + (\mathrm{Div}\,\xi)\mathcal{L} = \mathrm{Div}\,B$ (see [31] for details); dealing with evolutionary representatives, i.e. vertical vector fields, this reduces to $X^{(1)}[\mathcal{L}] = \mathrm{Div}\,B \equiv D_i B^i$. We say that $X = \varphi^a(\partial/\partial u^a)$ is a *divergence μ-symmetry* for the Lagrangian $\mathcal{L}$ of order r if there exists a (matrix-valued) p-tuple $\mathcal{B}$ such that

$$Y[\mathcal{L}] = (\nabla_i)^a{}_b (\mathcal{B}^i)^b{}_a = \mathrm{Tr}[\nabla_i \mathcal{B}^i]. \tag{28}$$

Theorem 3. *The vector field $X = \varphi^a(\partial/\partial u^a)$ is a divergence μ-symmetry for the first-order Lagrangian $\mathcal{L}(u, u_x)$ if and only if there is $\mathcal{B}$ such that, with $\mathcal{P}^a{}_b := \varphi^a \pi_b$, $(\mathcal{P} - \mathcal{B})$ is a μ-conserved vector.*

Proof. Let us assume that (28), with $r = 1$, is satisfied. As seen in the proof to theorem 1, $X^{(1)}_{(\mu)}[\mathcal{L}] = \left(\nabla^a{}_b\right)_i \left(\pi^i_a \varphi^b\right)$; hence, (28) implies

$$\left(\nabla^a{}_b\right)_i \left(\pi^i_a \varphi^b\right) = \left(\nabla^a{}_b\right)_i (\mathcal{B}^i)^b{}_a. \tag{29}$$

The term on the left-hand side is just $\nabla_i \mathcal{P}^i$, and therefore (29) reads

$$\left(\nabla^a{}_b\right)_i \left[(\mathcal{P}^i)^b{}_a - (\mathcal{B}^i)^b{}_a\right] = 0. \tag{30}$$

Going through this computation the other way round, we also show that μ-conservation of $(\mathcal{P} - \mathcal{B})$ implies $X^{(1)}_{(\mu)}[\mathcal{L}] = \mathrm{Tr}(\nabla_i \mathcal{B}^i)$. $\qquad\square$

As anticipated, the above result allows us to include in our discussion the case of evolutionary vector fields and of generalized vector fields as well.

Theorem 4. *The first-order Lagrangian $\mathcal{L}$ is μ-invariant under the vector field X (now possibly with $\xi^i \neq 0$) written in evolutionary form as $X = Q^a(\partial/\partial u^a)$ if and only if the quantity*

$$(\mathcal{P}^i - \mathcal{B}^i)^b{}_a \equiv Q^b \frac{\partial \mathcal{L}}{\partial u_i^a} + \mathcal{L}\xi^i \delta^b{}_a$$

is μ-conserved, i.e. if

$$\mathrm{Tr}(\nabla^i(\mathcal{P}_i - \mathcal{B}_i)) \equiv D^i \mathbf{P}_i + D^i(\mathcal{L}\xi_i) + (\Lambda_i Q)^a \frac{\partial \mathcal{L}}{\partial u_i^a} + \mathcal{L}\,\mathrm{Tr}(\Lambda_i \xi^i) = 0.$$

Proof. The condition of μ-invariance for a first-order Lagrangian under a generic vector field in evolutionary form $X_Q = Q^a(\partial/\partial u^a)$ can be written as

$$X^{(1)}_{(\mu)}[\mathcal{L}] = (\nabla^i)^a{}_b \, (\mathcal{B}_i)^b{}_a \qquad \text{where} \quad (\mathcal{B}_i)^b{}_a = -\mathcal{L}\xi_i \delta^b{}_a, \tag{31}$$

as in (30). $\qquad\square$

Remark 1. It should be recalled that in the standard case, divergence symmetries of the Lagrangian are also symmetries of the corresponding Euler–Lagrange equations. This is in general *not* the case for divergence μ-symmetries, as already discussed and pointed out in [27] for the case of λ-symmetry. See also the subsection 6.3.

Remark 2. We stress that the functional form of the M-vector $\mathcal{P}$ is the same as that of the standard vector $\mathbf{P}$ met in standard Noether theorem. In particular, if $\xi^i \equiv 0$, i.e. for $Q = Q(u) = \varphi(u)$, and hence Q not depending on the momenta, $\mathcal{P}$ is homogeneous of order one in the momenta; allowing generalized symmetries with Q depending on momenta, the $\mathcal{P}$ would not be homogeneous of order one (we could thus have conservation laws of higher order in the momenta; see [21, 33] for a discussion of this case and relevant bibliography) but would however always depend on momenta. Also in this more general case, all things are completely analogous to those arising in the same context in the framework of standard Noether theory and will not be discussed here.

5. Conditional invariants and μ-symmetries

Special cases of the conservation law (19) for first-order Lagrangians can occur depending on the form of the rhs of the equation. For instance, the case where $\Lambda_i \varphi = \lambda_i \varphi$ (with $p > 1$) will be considered in subsections 6.3 and 6.4.

A particularly important case occurs restricting to a single (i.e. $p = 1$) independent variable; we will then switch to a 'mechanical' notation, denoting this as t and the dependent variables as $q^a = q^a(t)$, with $dq^a/dt := \dot{q}^a$. Now $\mathcal{P}$ is a single matrix, and the same is for Λ.

Let us recall that, considering first-order Lagrangians, writing $p_a := \partial\mathcal{L}/\partial\dot{q}^a$, if there is some function $\alpha(q, \dot{q})$ such that $X^{(1)}[\mathcal{L}] = [\alpha(q, \dot{q})](\varphi^a p_a)$ (recall that $X^{(1)}$ is the standard first prolongation of X), we say that X is a *conditional* symmetry of $\mathcal{L}$: it is indeed a symmetry

of the Lagrangian restricted to the subspace $\mathbf{P} = \varphi^a p_a = 0$. Then $\mathbf{P}$ is a *conditionally conserved* quantity, i.e. the relation $\mathbf{P} = 0$ (or, which is the same, the zero-level manifold for $\mathbf{P}$) is preserved under the motion generated by $\mathcal{L}$: in fact, $D_t \mathbf{P} = X^{(1)}[\mathcal{L}] = \alpha \mathbf{P}$. One speaks then of a *conditional invariant* [7, 32, 36], or also of an *invariant relation* in the sense of Levi-Civita [1, 22]. This admits an obvious interpretation and extension to μ-symmetries of the Lagrangian.

Theorem 5. *Let* $X = \varphi^a(\partial/\partial q^a)$, $\mathcal{L}$ *be a first-order Lagrangian,* $\mu = \Lambda dt$ *and assume* $\Lambda \varphi = \lambda \varphi$ *where* $\lambda = \lambda(t, q, \dot{q})$ *is a scalar function. With* $\mathcal{P}_a{}^b := p_a \varphi^b$, $\mathbf{P} = \operatorname{Tr} \mathcal{P}$, *if one has* $X^{(1)}_{(\mu)}[\mathcal{L}] = \alpha(q, \dot{q}) \mathbf{P}$, *for some scalar function* α, *then*

$$D_t \mathbf{P} = (\alpha - \lambda) \mathbf{P} \tag{32}$$

and $\mathbf{P}$ *is a conditionally* μ-*invariant quantity.*

Proof. The proof follows immediately from the calculations in section 3, which show in particular that

$$X^{(1)}_{(\mu)}[\mathcal{L}] = (\nabla_t)^b{}_a(\mathcal{P})^a{}_b = D_t \mathbf{P} + \lambda \varphi^a(\partial \mathcal{L}/\partial \dot{q}^a)$$

and we reduce to (32) in our hypotheses. □

Alternatively, X may be interpreted as a μ-symmetry for the Lagrangian but now with $\lambda' := \lambda - \alpha$. In other words, $\mathbf{P}$ is both μ-conserved (i.e. $\nabla_t \mathbf{P} = (D_t + \lambda')\mathbf{P} = 0$) and conditionally conserved (i.e. $D_t \mathbf{P} = 0$ on the manifold $\mathbf{P} = 0$).

By the way, the result can be extended to the general case with more than one independent variable ($p > 1$). Indeed, coming back to the previous notation $u^a(x^i)$, let us assume that, given a vector field X and a form $\mu = \Lambda^i dx_i$, a first-order Lagrangian $\mathcal{L}$ satisfies

$$X^{(1)}_{(\mu)}[\mathcal{L}] = \operatorname{Tr}(A_i \mathcal{P}^i)$$

for some p matrices A_i: then we get the conditional μ-conservation law

$$D_i \mathbf{P}^i = \operatorname{Tr}((A_i - \Lambda_i)\mathcal{P}^i).$$

6. Gauge equivalence and μ-conservation laws

It was shown in [10] that given any vector field X in $\mathcal{M}$, the μ-prolonged vector field $X^{(r)}_{(\mu)}$ in $J^r \mathcal{M}$ is locally gauge-equivalent (in a sense recalled below) to a vector field W in $J^r \mathcal{M}$ which is the *standard* prolongation of some vector field $\widetilde{X}$ in $\mathcal{M}$; moreover, X is locally gauge-equivalent to $\widetilde{X}$ via the same gauge transformation[5].

6.1. Gauge equivalence

The gauge transformations mentioned here act in this way: there is a linear representation T_g of the gauge group[6] G acting in the space $U = \mathbf{R}^q$ of the dependent variables, hence in the fibers of $(\mathcal{M}, \pi, \mathcal{B})$. This extends to a representation of G in the fibers of the jet spaces $J^r \mathcal{M}$

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[5] Strictly speaking, this is true for PDEs; when dealing with ODEs other possibilities appear, as discussed in [10, 24, 34], and one can end up dealing with non-local vector fields of exponential type. In this paper we will however disregard these possibilities.

[6] As traditional in physical literature, we denote as 'gauge group' both the finite-dimensional group G which acts on the space U, i.e. on the fibers of the bundle $(\mathcal{M}, \pi, \mathcal{B})$, and the group of gauge transformations, i.e. of sections Γ of the relevant principal bundle P_G with fiber G [28, 29]; in this case P_G is a bundle over $\mathcal{M}$ (see [13]), and an element $\gamma \in \Gamma$ is a function $\gamma(x, u)$ taking values in G.

of any order, acting in each of the $U_J \simeq \mathbf{R}^q$ spaces corresponding to variables $\{u^1_J, \ldots, u^q_J\}$ with the same multi-index J (this was denoted as a *jet representation* in [10]).

Then, if $\gamma : \mathcal{M} \to G$ is an element of the gauge group, i.e. a function from $\mathcal{M}$ to the local gauge group G, it defines a matrix-valued function $A_\gamma(x, u)$ via $A_\gamma(x, u) = T_{\gamma(x,u)}$; that is, at $(x, u) \in \mathcal{M}$ we have the matrix representing the element $g = \gamma(x, u) \in G$. The gauge transformation γ acts in U_J by these matrices; note that the action on vector fields will then be described by the push-forward of this map, given (with D the differential) by $R_\gamma = A^{-1}_\gamma(DA_\gamma)$.

In the following we will, with an abuse of notation (already used in [10]), simply write γ for the R_γ obtained in this way.

Thus, if a vector field X is described in components by $X = \varphi^a(\partial/\partial u^a)$, the gauge transformed via γ will be

$$\widetilde{X} = \widetilde{\varphi}^a(\partial/\partial u^a) = (\gamma\varphi)^a(\partial/\partial u^a) = \left(\gamma^a{}_b\,\varphi^b\right)(\partial/\partial u^a). \tag{33}$$

The results recalled above [10] guarantee that the vector field $\widetilde{X}$ and its standard prolongation W are gauge equivalent respectively to X and to its μ-prolongation $Y = X_{(\mu)}$. We write

$$\widetilde{X} = \gamma \cdot X, \qquad W = \gamma \cdot Y, \qquad \text{with} \qquad Y = \psi^a_J \frac{\partial}{\partial u^a_J}, \qquad W = \widetilde{\varphi}^a_J \frac{\partial}{\partial u^a_J}, \tag{34}$$

one has then (see [10])

$$\psi^a_J = \gamma^{-1}\widetilde{\varphi}^a_J. \tag{35}$$

The setting of μ-symmetries is recovered by writing

$$\Lambda_i = \gamma^{-1}D_i\gamma; \tag{36}$$

in this case (6) is automatically satisfied, and conversely (6) guarantees (for $q > 1$) that locally Λ_i can be written in the form (36) [10].

Suppose now we have a vector field $Y = X^{(r)}_{(\mu)}$ such that $Y[\mathcal{L}] = 0$. That is, X is a μ-symmetry for $\mathcal{L}$ and hence, by theorem 1, there is a μ-conservation law associated with it. One could expect that this implies $\widetilde{X}[\mathcal{L}] = 0$, hence that $\widetilde{X}$ would be a standard symmetry of $\mathcal{L}$, and by (standard) Noether theorem that there would be a standard conservation law associated with this symmetry. In the next subsections, we shall show that things are not like this in such a generality.

6.2. The general case

Let us assume that a Lagrangian $\mathcal{L}$ is μ-invariant under some vector field X, i.e. $X^{(r)}_{(\mu)}[\mathcal{L}] = Y[\mathcal{L}] = 0$, and therefore that the μ-conservation law

$$\left(\delta^a{}_b\,D_i + (\Lambda_i)^a{}_b\right)\left[(\mathcal{P}^i)^a{}_b\right] = 0 \tag{37}$$

holds. On the other hand, using (36), the lhs of (37) is transformed into

$$(D_i\mathcal{P}^i)^a{}_a + (\Lambda_i)^a{}_b\,(\mathcal{P}^i)^b{}_a = D_i\,\mathbf{P}^i + (\gamma^{-1})^a{}_c(D_i\gamma)^c{}_b(\mathcal{P}^i)^b{}_a$$
$$= (\gamma^{-1})^a{}_c\left[\left(\gamma^c{}_b D_i + (D^i\gamma)^c{}_b\right)(\mathcal{P}^i)^b{}_a\right] = (\gamma^{-1})^a{}_c D_i\left(\gamma^c{}_b(\mathcal{P}^i)^b{}_a\right). \tag{38}$$

Thus, the μ-conservation law can be rewritten, in terms of the matrix γ, in the very general and compact form

$$\mathrm{Tr}[\gamma^{-1}D_i(\gamma\mathcal{P}^i)] = 0. \tag{39}$$

In particular, this shows that in general no standard conservation law is associated with the considered symmetry. A significant exception to this conclusion is provided by the case $q = 1$, $p > 1$, as discussed below.

6.3. The case of a scalar field

In the particular case $q = 1$ (that is, a single unknown function $u = u(x)$ of $p > 1$ variables $\{x^1, \ldots, x^p\}$; in physical terms, a scalar field), the p matrices Λ_i are actually p functions λ_i, and from the condition (5) one deduces the existence (at least locally, see [10]) of a nonvanishing scalar function $\gamma = \gamma(x, u)$ such that

$$\lambda_i = \gamma^{-1} D_i \gamma. \tag{40}$$

Theorem 6. *If $q = 1$, $p > 1$, and a Lagrangian is μ-invariant under the vector field X, then the conservation law can be expressed in the* standard *form*

$$D_i \widetilde{\mathbf{P}}^i = 0 \tag{41}$$

where $\widetilde{\mathbf{P}}^i$ is the 'current density vector' determined by the vector field $\widetilde{X} := \gamma X$.

Proof. This easily follows from the above general formula when there is only one component ($q = 1$) for the considered vectors. More specifically, recalling that if $X = \varphi(\partial/\partial u)$ is a μ-symmetry, then there is in this case an equivalent standard symmetry $\widetilde{X} := \gamma X = (\gamma \varphi)(\partial/\partial u)$, see [10] and the discussion above: one has indeed $\widetilde{X}^{(r)} = \gamma X^{(r)}_{(\mu)}$ for all orders of prolongations. $\qquad \square$

Let us consider in explicit form, for instance, the case of first-order prolongation and first-order Lagrangians.

For the first-order μ-prolongation one has

$$X^{(1)}_{(\mu)} = \varphi \frac{\partial}{\partial u} + (D_i \varphi + \lambda_i \varphi) \frac{\partial}{\partial u_i} = \gamma^{-1} \left(\gamma \varphi \frac{\partial}{\partial u} + D_i(\gamma \varphi) \frac{\partial}{\partial u_i} \right) = \gamma^{-1} \widetilde{X}^{(1)};$$

if $\mathcal{L}$ is a first-order Lagrangian, then the μ-invariance condition reads

$$X^{(1)}_{(\mu)}[\mathcal{L}] = \varphi \frac{\partial \mathcal{L}}{\partial u} + (D_i \varphi + \lambda_i \varphi) \frac{\partial \mathcal{L}}{\partial u_i}$$
$$= \varphi \frac{\partial \mathcal{L}}{\partial u} + \gamma^{-1} D_i \left(\gamma \varphi \frac{\partial \mathcal{L}}{\partial u_i} \right) - \varphi D_i \frac{\partial \mathcal{L}}{\partial u_i} = \varphi \mathcal{E}[\mathcal{L}] + \gamma^{-1} D_i \left(\gamma \varphi \frac{\partial \mathcal{L}}{\partial u^i} \right) = 0$$

and the conservation law can be given in the standard form, as stated by theorem 6, $D_i \widetilde{\mathbf{P}}^i \equiv D_i(\gamma \varphi \pi^i) = 0$.

6.4. Conservation laws for multi-component fields

Theorem 6 holds for $q = 1$, and the discussion in subsection 6.2 shows that for $q > 1$ it is in general not possible to obtain conservation laws in standard form from μ-symmetries.

Another exception occurs in this case if, considering a Lagrangian of order r, the matrices Λ_i satisfy the condition

$$\Lambda_i (D_J \varphi^a) = \lambda_i (D_J \varphi^a) \qquad \forall J \quad \text{with} \quad |J| \leqslant r - 1, \tag{42}$$

where D_J is the total derivative with respect to the multi-index J of order $|J|$, and λ_i are scalar functions.

In this case (mentioned also in subsection 2.3), a conservation law as in (41) still holds. Rather than discussing the general case, we will limit to first-order Lagrangians; for these ($r = 1$) condition (42) is simply

$$(\Lambda_i \varphi)^a \equiv (\Lambda_i)^a{}_b \varphi^b = \lambda_i \varphi^a. \tag{43}$$

Theorem 7. *Let $\mu = \Lambda_i \mathrm{d}x^i$ satisfy equation (6), and $\mathcal{L}$ be a first-order Lagrangian admitting as μ-symmetry a vector field $X = \varphi^a (\partial/\partial u^a)$ such that (43) is satisfied; let γ be defined as in (40). Then the standard conservation law holds*

$$D_i \left(\gamma \varphi^a \frac{\partial \mathcal{L}}{\partial u_i^a} \right) = 0. \tag{44}$$

Proof. The argument used in theorem 6 works also in this case, thanks to (43). $\qquad\square$

Using the same arguments, the result can also be suitably extended to the case of divergence μ-symmetries.

Theorem 8. *Assume either $q = 1$, $p > 1$; or that (43) is satisfied if $q > 1$. Let $\mathcal{L}$ be a first-order Lagrangian, X a vector field, $\mu = \Lambda_i \mathrm{d}x^i$ a form satisfying the equation (6), with γ defined as in (40) and $\mathbf{P}^i = \varphi^a \pi_a^i$ the usual current density vector.*

(a) *If there is a $p-$ tuple B^i such that $X_{(\mu)}^{(1)}[\mathcal{L}] = \gamma^{-1}\mathrm{Div}\, B$, then the conservation law in standard form $D_i(\gamma \mathbf{P}^i - B^i) = 0$ holds.*

(b) *If there is a p-tuple B^i such that $X_{(\mu)}^{(1)}[\mathcal{L}] = \lambda_i B^i + \mathrm{Div}\, B$, then the conservation law in standard form $D_i(\gamma \mathbf{P}^i - \gamma B^i) = 0$ holds.*

Let us finally remark that clearly if X is a μ-symmetry (or also a divergence μ-symmetry) for a Lagrangian, then the corresponding Euler–Lagrange equation turns out in this case to be symmetric with respect to the gauge-equivalent standard symmetry $\widetilde{X} = \gamma X$.

7. Variational problems with μ-prolonged variation field

In this section, we reverse our point of view, by the introduction of the notion of *modified μ-Euler–Lagrange equations*.

The Euler–Lagrange equations are indeed obtained requiring that the action integral $S = \int \mathcal{L}\, \mathrm{d}^p x$ is stationary under a variation δu (vanishing on the frontier ∂B of the region of integration) of the dependent variables $u^a(x)$. The variation field $V = (\delta u)^a (\partial/\partial u^a)$ induces a variation of the u_i^a as well; standard Euler–Lagrange equations (that we have considered so far) are obtained in the case where this variation is described by the standard lift of the variation δu, i.e. by the standard prolongation of V.

We will now consider the case where the variation field is μ-prolonged to obtain the variation in the space of field derivatives.

7.1. Derivation of the μ-Euler–Lagrange equations

Let us consider a first-order Lagrangian $\mathcal{L}(u, u_x)$, and the action integral

$$S(A) = \int_A \mathcal{L}(u, u_x)\, \mathrm{d}^p x.$$

We consider the variation $u \to u + \delta u$ corresponding to the action of a vector field $V = \eta^a(x, u)(\partial/\partial u^a)$; that is, $\delta u^a = \varepsilon \eta^a(x, u)$; needless to say this also acts on the u_i^a.

We consider the case where this action on field derivatives is described by the μ-prolongation of V, i.e. by

$$V_\mu^{(1)} = \eta^a \frac{\partial}{\partial u^a} + \left[D_i \eta^a + (\Lambda_i)^a{}_b\, \eta^b \right] \frac{\partial}{\partial u_i^a};$$

this corresponds to

$$\delta u^a = \varepsilon \eta^a, \qquad \delta u_i^a = \varepsilon (\nabla_i)^a{}_b \, \eta^b. \tag{45}$$

The equations satisfied by fields $\overline{u}$ such that the variation $(\delta S(A))[u]$ of S under (45) vanishes at $u = \overline{u}$ for all variations δu satisfying $[\delta u]_{\partial A} = 0$, will be called the μ-*Euler–Lagrange equations* and are derived in the same way as the standard Euler–Lagrange equations.

The variation of S under (45) is given by

$$\delta S = \int_A (\delta L) \, \mathrm{d}^p x = \int_A \left[\frac{\partial \mathcal{L}}{\partial u^a} \delta u^a + \frac{\partial \mathcal{L}}{\partial u_i^a} \delta u_i^a \right] \mathrm{d}^p x. \tag{46}$$

According to (45), $\delta u_i^a = D_i(\delta u^a) + (\Lambda_i)^a{}_b \, \delta u^b$; substituting this into (46) and integrating by parts, we have

$$\delta S = \int_{\partial A} \frac{\partial \mathcal{L}}{\partial u_i^a} \delta u^a \, \mathrm{d}\sigma + \int_A \left[\left(\frac{\partial \mathcal{L}}{\partial u^a} - D_i \frac{\partial \mathcal{L}}{\partial u_i^a} \right) \delta u^a + \frac{\partial \mathcal{L}}{\partial u_i^a} (\Lambda_i)^a{}_b \, \delta u^b \right] \mathrm{d}^p x$$

where $\mathrm{d}\sigma$ is the (surface) element of integration on ∂A. The boundary term vanishes (due to $\delta u = 0$ on ∂A), and rearranging indices we get hence

$$\delta S = \int_A \left[\frac{\partial \mathcal{L}}{\partial u^a} - D_i \frac{\partial \mathcal{L}}{\partial u_i^a} + \frac{\partial \mathcal{L}}{\partial u_i^b} (\Lambda_i)^b{}_a (\delta u^a) \right] \mathrm{d}^p x.$$

As we require this to be zero for any choice of δu^a, the term in square brackets has to vanish, i.e. we have established the μ-Euler–Lagrange equations to be

$$\frac{\partial \mathcal{L}}{\partial u^a} - D_i \frac{\partial \mathcal{L}}{\partial u_i^a} + \frac{\partial \mathcal{L}}{\partial u_i^b} (\Lambda_i)^b{}_a = 0. \tag{47}$$

Remark 3. Writing Λ_i^T for the transpose of Λ_i, and (as above) $\pi_a^i := \left(\partial \mathcal{L} / \partial u_i^a \right)$ for the momenta, the (47) are also written as

$$(\partial \mathcal{L} / \partial u^a) - D_i \pi_a^i = -\left(\Lambda_i^T \right)_a{}^b \pi_b^i.$$

Remark 4. It would not be difficult, proceeding along these lines in the same way as in the standard case (i.e. by repeated integration by parts), to obtain the μ-Euler–Lagrange equations for higher order Lagrangians.

7.2. Conservation laws from μ-symmetries

We will now consider the case where $\mathcal{L}$ admits a μ-symmetry $X = \varphi^a (\partial / \partial u^a)$, with μ the same form appearing in the μ-prolongation of the variation field; we will as usual denote by Y its μ-prolongation, $Y = \varphi^a (\partial / \partial u^a) + \left[(\nabla_i)^a{}_b \varphi^b \right] (\partial / \partial u_i^a)$, so that our assumption is that $Y[\mathcal{L}] = 0$. Recalling also $(\nabla_i)^a{}_b = \delta_b^a D_i + (\Lambda_i)^a{}_b$, the symmetry condition reads[7]

$$\varphi^a \frac{\partial \mathcal{L}}{\partial u^a} + \left(D_i \varphi^a + (\Lambda_i)^a{}_b \varphi^b \right) \frac{\partial \mathcal{L}}{\partial u_i^a} = 0. \tag{48}$$

Theorem 9. *Let $\mathcal{L}$ be a first-order Lagrangian, admitting the vector field $X = \varphi^a (\partial / \partial u^a)$ as a μ-symmetry for a certain form μ. Then the vector $\mathbf{P}$ of components $\mathbf{P}^i = \varphi^a \pi_a^i$ defines a standard conservation law, $D_i \mathbf{P}^i = 0$, for the flow of the associated μ-Euler–Lagrange equations.*

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[7] This is of course the same as the condition met above, see e.g. theorem 1: the symmetry vector field and the Lagrangian are the same as in the standard variation case, the difference residing only in considering the (standard or instead μ-) Euler–Lagrange equation.

Proof. With an integration by parts, (48) is transformed into

$$\varphi^a \frac{\partial \mathcal{L}}{\partial u^a} + D_i \left(\varphi^a \frac{\partial \mathcal{L}}{\partial u_i^a} \right) - \varphi^a D_i \left(\frac{\partial \mathcal{L}}{\partial u_i^a} \right) + \left((\Lambda_i)^b{}_a \varphi^a \right) \frac{\partial \mathcal{L}}{\partial u_i^b} = 0.$$

(We have exchanged indices in the last term—which is fine since they are both summation indices.) Collecting the terms with φ^a, we get this is in the form

$$\varphi^a \left[\frac{\partial \mathcal{L}}{\partial u^a} - D_i \left(\frac{\partial \mathcal{L}}{\partial u_i^a} \right) + \frac{\partial \mathcal{L}}{\partial u_i^b} (\Lambda_i)^b{}_a \right] + D_i \left(\varphi^a \frac{\partial \mathcal{L}}{\partial u_i^a} \right) = 0.$$

The term in square brackets vanishes due to (47); hence, (48) implies the conservation law

$$D_i \left(\varphi^a \frac{\partial \mathcal{L}}{\partial u_i^a} \right) = 0, \tag{49}$$

as claimed in the statement. □

7.3. Discussion

We stress that (49) is a standard conservation law, whose conserved vector **P** has the familiar form of conserved vector for the standard Noether theorem. Hence a μ-symmetry is for the μ-variational equations what a standard symmetry is for the standard variational equations.

   This should not be surprising, in view of the discussion in section 6. In fact, now both the variation vector fields $V_{(\mu)}$ and the symmetry vector field $Y = X_{(\mu)}$ are μ-prolongations (with the same μ) of Lie-point vector fields in $\mathcal{M}$, and according to [10] they could be taken to be standard prolongations of (different) Lie-point vector fields via a gauge transformation. With reference to the arbitrariness of the variation vector field, it is worth noting that any gauge transformation would transform vector fields in $\mathcal{M}$ vanishing on a set $\partial A \subset \mathcal{M}$ into (generally, different) vector fields also vanishing on the same set.

   Thus, the case of μ-symmetries for a Lagrangian $\mathcal{L}$ and evolution law described by μ-Euler–Lagrange equations for $\mathcal{L}$ should be seen (locally, see [10]) as a gauge transformation of the situation where $\mathcal{L}$ has a standard symmetry and the evolution law is given by the standard Euler–Lagrange equations.

   As suggested by this correspondence, the situation where we have an evolution controlled by the μ-Euler–Lagrange equations (47) and a standard symmetry is symmetric (or, maybe more precisely, dual) to the situation studied in sections 3–6, and would produce a $\widetilde{\mu}$-conservation law (we use the notation $\widetilde{\mu}$ to stress this is a μ-conservation law with a different μ; actually: $\widetilde{\mu} = -\mu$).

   In fact, consider the μ-Euler–Lagrange equations (47) for a Lagrangian $\mathcal{L}$ and a form $\mu = \mu_0 = \Lambda_i \, dx^i$; and assume this admits the vector field $X = \varphi^a (\partial / \partial u^a)$ as a standard symmetry, so that $X^{(1)}[\mathcal{L}] = 0$. Recalling the (standard) prolongation law, this means

$$\varphi^a \left(\frac{\partial \mathcal{L}}{\partial u^a} \right) + (D_i \varphi^a) \left(\frac{\partial \mathcal{L}}{\partial u_i^a} \right) \equiv \varphi^a \left[\frac{\partial \mathcal{L}}{\partial u^a} - D_i \left(\frac{\partial \mathcal{L}}{\partial u_i^a} \right) \right] + D_i \left(\varphi^a \pi_a^i \right) = 0.$$

Using (47), this reads

$$D_i \left(\varphi^a \pi_a^i \right) = \frac{\partial \mathcal{L}}{\partial u_i^b} (\Lambda_i)^b{}_a \varphi^a,$$

which is indeed a μ-conservation law, see section 2, for $\mu = -\Lambda_i \, dx^i = -\mu_0$.

8. Applications and examples

We will now discuss briefly some general aspects of μ-symmetries and μ-conservation laws, together with some of their possible interpretation and applications.

First of all, as already pointed out, one of the main properties coming from the presence of a μ-symmetry of a given Lagrangian is that the ordinary Noether theorem is replaced by a suitable modification of the conservation law. It is clear, on the other hand, that the existence of a conservation law (even in a weakened or modified form) can be *per se* a relevant result.

More specifically, in some cases, the μ-symmetry can be obtained as a perturbation of an exact symmetry, and—accordingly—the μ-conservation as a perturbation of the standard Noether law. This will be illustrated by some examples below, see in particular examples 1, 5 and 6. Example 1 will be also considered in the setting of section 7, i.e. by the introduction of the μ-modified Euler–Lagrange equations.

More generally, consider a—say, first-order—Lagrangian $\mathcal{L}_0$ (and/or second-order equations $\mathcal{E}_0$, possibly the Euler–Lagrangian equations for $\mathcal{L}_0$) which are symmetric under a vector field X, i.e. under its first (respectively, second) standard prolongation. Consider now a perturbation $\mathcal{L}_\varepsilon = \mathcal{L}_0 + \varepsilon\mathcal{L}_1 + \varepsilon^2\mathcal{L}_2 + \cdots$ (and/or $\mathcal{E}_\varepsilon = \mathcal{E}_0 + \varepsilon\mathcal{E}_1 + \varepsilon^2\mathcal{E}_2 + \cdots$); we wonder if this may admit X as a μ-symmetry with nontrivial μ (as we assumed X is not a standard symmetry of $\mathcal{L}_\varepsilon$ or $\mathcal{E}_\varepsilon$).

We have shown that in some cases the μ-conservation law actually reduces to a standard one, and also pointed out the existence of a relation with conditional invariants: see examples 4, 6 and 7.

Another important aspect related to the presence of μ-symmetries is the possibility of reducing the order of differential equations. In the case of variational problems, the first difficulty—as well known [27]—is that the Euler–Lagrange equations do not inherit, in general, the μ-invariance of the Lagrangian; this is clearly a strong difference with respect to the case of exact symmetries. As a consequence, only a 'partial' reduction can be expected, along the same lines pointed out in [27]. Some aspects of this nontrivial problem will be illustrated in examples 3, 4 and 7.

One of the differences between standard conservation law and a μ-conservation law lies in that a standard conservation law allows for an algebraic substitution eliminating one of the momenta (or variables) in the Euler–Lagrange equations, whereas a μ-conservation law amounts to an auxiliary differential equation which has to be solved in order to implement the reduction allowed by the conservation law itself. This will be useful if the auxiliary equation is easier to solve (in general, or at least in order to provide some special solution) than the Euler–Lagrange equations.

In some cases, as shown in examples 8 and 9, it can be possible to explicitly integrate the μ-conservation law: this may help to obtain solutions, or at least to suggest how to find particular solutions to the Euler–Lagrangian equations.

We finally refer to [13] for a different interpretation of μ-symmetries in terms of changes of frame, or better of (pointwise linear) transformations acting in the same way on the vertical vector spaces for the fibrations $J^{k+1}M \to J^k M$ ($k \geqslant 0$).

We are now proposing some examples illustrating the above discussion; some of the examples are, admittedly, rather artificial, but they can be useful to clarify also more technical details of the procedure.

In the examples below, when the Lagrangian involves two functions depending on two variables, we shall write, to simplify notations, x, y instead of x^1, x^2, and $u = u(x, y), v = v(x, y)$ instead of u^1, u^2. If there is only one independent variable, we will use the same

'mechanical' notation $q^a(t)$ as in section 5 (but, in order to avoid any confusion with powers, we will actually adopt the notation with lower indices $q_1(t), q_2(t)$).

Example 1. Consider the vector field

$$X = u\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \tag{50}$$

where $u = u(x, y)$, $v = v(x, y)$ and with the form $\mu = \Lambda_1\,dx + \Lambda_2\,dy$ defined by

$$\Lambda_1 = \begin{pmatrix} 0 & 0 \\ u_x & 0 \end{pmatrix} \qquad \Lambda_2 = \begin{pmatrix} 0 & 0 \\ u_y & 0 \end{pmatrix}.$$

It is easy to check that the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(u_x^2 + u_y^2\right) - \frac{1}{u}(u_x v_x + u_y v_y) + u^2 \exp(-2v)$$

is μ-invariant (but not invariant) under the vector field X. The M-vector $(\mathcal{P}^i)^b{}_a$ is explicitly given by

$$\left((\mathcal{P}^1)^b{}_a, (\mathcal{P}^2)^b{}_a\right) = \left(\begin{pmatrix} uu_x - v_x & -u_x \\ u_x - v_x/v & -u_x/u \end{pmatrix}, \begin{pmatrix} uu_y - v_y & -u_y \\ u_y - v_y/v & -u_y/u \end{pmatrix}\right);$$

it satisfies the μ-conservation law $\mathrm{Tr}(\nabla_i\mathcal{P}^i) = 0$, which takes here the form

$$D_i\mathbf{P}^i \equiv D_x\left(uu_x - v_x - \frac{u_x}{u}\right) + D_y\left(uu_y - v_y - \frac{u_y}{u}\right) = u_x^2 + u_y^2$$

where $\mathbf{P}^i = \mathrm{Tr}(\mathcal{P}^i)$, using also the Euler–Lagrange equations, or

$$u(1 - u^2)(u_{xx} + u_{yy}) + u^2(v_{xx} + v_{yy}) = u_x^2 + u_y^2.$$

In agreement with theorem 1, the rhs of this expression is precisely equal to $-\mathrm{Tr}(\Lambda_i\mathcal{P}^i) = -(\Lambda_i\varphi)^a\frac{\partial\mathcal{L}}{\partial u_i^a}$. Note that the quantity $u_x^2 + u_y^2$ is just the 'symmetry-breaking term', i.e. the term which prevents the above Lagrangian from being exactly symmetric under the vector field (50).

Similar results hold for any Lagrangian depending, more in general, on the quantities x, y, $z_1 = e^{-v}u$, $z_2 = e^{-v}u_x$, $w_1 = u_x^2/2 - (u_x v_x)/u$, $w_2 = u_y^2/2 - (u_y v_y)/u$.

Example 2. Let us consider the same setting and notation of example 1, but we adopt here the point of view of section 7. The modified μ-Euler–Lagrange equations (47), for a generic first-order Lagrangian, read

$$D_x\left(\frac{\partial\mathcal{L}}{\partial u_x}\right) + D_y\left(\frac{\partial\mathcal{L}}{\partial u_y}\right) = \frac{\partial\mathcal{L}}{\partial u} + u_x\frac{\partial\mathcal{L}}{\partial v_x} + u_y\frac{\partial\mathcal{L}}{\partial v_y} \tag{51}$$

$$D_x\left(\frac{\partial\mathcal{L}}{\partial v_x}\right) + D_y\left(\frac{\partial\mathcal{L}}{\partial v_y}\right) = \frac{\partial\mathcal{L}}{\partial v}. \tag{52}$$

In the present case $\mathbf{P}$ is the vector of components

$$\mathbf{P} = \left(u\frac{\partial\mathcal{L}}{\partial u_x} + \frac{\partial\mathcal{L}}{\partial v_x}, u\frac{\partial\mathcal{L}}{\partial u_y} + \frac{\partial\mathcal{L}}{\partial v_y}\right)$$

and the conservation law claimed by theorem 9 will be given by the vanishing of

$$D_i\mathbf{P}^i = u_x\frac{\partial\mathcal{L}}{\partial u_x} + uD_x\frac{\partial\mathcal{L}}{\partial u_x} + D_x\frac{\partial\mathcal{L}}{\partial v_x} + u_y\frac{\partial\mathcal{L}}{\partial u_y} + uD_y\frac{\partial\mathcal{L}}{\partial u_y} + D_y\frac{\partial\mathcal{L}}{\partial v_y}$$

$$= u\left(\frac{\partial\mathcal{L}}{\partial u} + u_x\frac{\partial\mathcal{L}}{\partial v_x} + u_y\frac{\partial\mathcal{L}}{\partial v_y}\right) + \frac{\partial\mathcal{L}}{\partial v} + u_x\frac{\partial\mathcal{L}}{\partial u_x} + u_y\frac{\partial\mathcal{L}}{\partial u_y}$$

having used (51) and (52). Let us now assume $\mathcal{L}$ admits X as a μ-symmetry: as said in example 1 this is the case if $\mathcal{L} = \mathcal{L}(x, y, z_1, z_2, w_1, w_2)$. It is easy to conclude, by direct calculation, that indeed $D_i \mathbf{P}^i \equiv 0$, as claimed by theorem 9.

Example 3. Consider the vector field, with $\xi_i \neq 0$,

$$X = x \frac{\partial}{\partial x} + \frac{\partial}{\partial v}$$

or, in evolutionary form,

$$X_Q = -x u_x \frac{\partial}{\partial u} + (1 - x v_x) \frac{\partial}{\partial v}$$

and the μ-form defined by the two matrices

$$\Lambda_1 = \Lambda_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

A Lagrangian satisfying the μ-invariance condition (31) is, for instance,

$$\mathcal{L} = \frac{1}{x} \left(u^2 + x^2 u_x^2 + u_y^2 + (x v u_x + v_y)^2 \right).$$

The vector $\mathbf{P}$ is

$$\mathbf{P} \equiv \left(-x u_x \frac{\partial \mathcal{L}}{\partial u_x}, \; -x u_x \frac{\partial \mathcal{L}}{\partial u_y} + (1 - x v_x) \frac{\partial \mathcal{L}}{\partial v_y} \right).$$

In the case of *exact* invariance of $\mathcal{L}$ under X, one would expect the conservation law $D_x \mathbf{P}^1 + D_y \mathbf{P}^2 + D_x(x\mathcal{L}) = 0$; in our case, the μ-invariance of $\mathcal{L}$ produces the following modified conservation law, in agreement with theorem 4,

$$D_x \mathbf{P}^1 + D_y \mathbf{P}^2 + D_x(x\mathcal{L}) = -(\Lambda_i Q)^a \frac{\partial \mathcal{L}}{\partial u_i^a} = x u_x \frac{\partial \mathcal{L}}{\partial v_y} = 2 \left(x v u_x^2 + u_x v_y \right).$$

It could be perhaps more interesting to look for the symmetry properties of the Euler–Lagrange equations for this Lagrangian. Let us recall indeed that the μ-invariance of the Lagrangian is in general not shared by the Euler–Lagrange equations; however, this does not exclude that one at least of these equations can be expressed in terms of μ-invariant quantities (and of derivatives thereof). This is precisely our case: indeed, introducing the quantities $\alpha = x u_x$, $\beta = x v u_x + v_y$, which are μ-invariant, i.e. $X_{(\mu)}^{(1)}(\alpha) = X_{(\mu)}^{(1)}(\beta) = 0$, one has that one of the Euler–Lagrange equations can be written as

$$\beta_y - \alpha\beta = 0.$$

This property is clearly of great help in finding explicit solutions; e.g., choosing the simplest possibility $\beta = 0$, it is easy to obtain for instance the particular solution $u = (\log x) \exp(y)$, $v = \exp(-e^y)$ of the Euler–Lagrange equations.

Example 4. We now consider a time-dependent problem for the two variables $q_1(t)$, $q_2(t)$; let

$$X = q_1 \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}$$

and the μ-form $\mu = \Lambda dt$, where $\Lambda = \lambda I$ with $\lambda = q_1$. A Lagrangian μ-invariant under X is for instance

$$\mathcal{L} = \frac{1}{2} \left(\frac{\dot{q}_1}{q_1} - q_1 \right)^2 + \frac{1}{2} (\dot{q}_2 - q_1)^2$$

and it is now easy to verify the existence of the μ-conservation law $\nabla_t \mathbf{P} = 0$ for the quantity $\mathbf{P} = (\dot{q}_1/q_1) + \dot{q}_2 - 2q_1$. This can also be interpreted as a conditional invariant relation

(cf section 5) in the form $D_t\mathbf{P} = X_\mu^{(1)}(\mathcal{L}) - \lambda\mathbf{P} = -q_1\mathbf{P}$. If we now introduce the μ-invariant quantities $\alpha = (\dot{q}_1/q_1) - q_1$, $\beta = \dot{q}_2 - q_1$, it is easy to see that the Euler–Lagrange equations (in this case a system of ODE's) can be written in the form

$$\alpha_t + q_1(\alpha + \beta) = 0, \qquad \beta_t = 0.$$

Then, as expected, they are not invariant under X, nor μ-invariant (or, better, only the second one of them is μ-invariant). However, this is a good example of a 'partial' reduction, in the same sense as explained in [27]: these equations indeed admit a reduction into the system of *first-order* equations $\alpha = \beta = 0$. This of course allows us to obtain easily the particular solution $q_1 = c_1/(1 - c_1 t)$, $q_2 = c_2 - \log(1 - c_1 t)$, where c_1, c_2 are arbitrary constants.

This also generalizes to a system of equations the result given in the above mentioned paper [27].

Example 5. We consider here a problem for the two variables $q_1(t), q_2(t)$ admitting as a μ-symmetry the rotation operator

$$X = q_2\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial q_2} = \frac{\partial}{\partial\theta}$$

having introduced as new variables the polar coordinates $r = r(t), \theta = \theta(t)$. Let us assume that μ is given by $\mu = \Lambda dt$ where, in the basis r, θ,

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon\cos\theta \end{pmatrix}$$

and ε is a 'small' real parameter. A μ-invariant Lagrangian is, e.g.,

$$\mathcal{L} = \tfrac{1}{2}(\dot{r}^2 + r^2(\dot{\theta} - \varepsilon\sin\theta)^2) - V(r)$$

where $V(r)$ is an arbitrary function. This Lagrangian can be viewed as a 'perturbation' (for $\varepsilon \neq 0$) of a rotationally-symmetric Lagrangian. It is simple to verify indeed that the angular momentum $r^2\dot{\theta}$ is not conserved; we have instead the μ-conservation law $D_t\mathbf{P} = (-\varepsilon\cos\theta)\mathbf{P}$ for the quantity

$$\mathbf{P} = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = r^2(\dot{\theta} - \varepsilon\sin\theta).$$

Example 6. As in the previous example, we consider the rotation vector field $X = \partial/\partial\theta$ as a (variational) μ-symmetry, but here the variables r, θ are the independent variables and a single dependent variable $u = u(r, \theta)$ is introduced; then $\varphi = 0, \xi_1 = 0, \xi_2 = 1$. We consider the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}r^2\exp(-\varepsilon\theta)u_r^2 + \tfrac{1}{2}\exp(\varepsilon\theta)u_\theta^2$$

which is clearly not invariant under rotation symmetry (if $\varepsilon \neq 0$), but is μ-invariant with $\mu = \varepsilon\,d\theta$. The above Lagrangian is the Lagrangian of a perturbed Laplace equation, indeed the Euler–Lagrange equation is the PDE

$$r^2 u_{rr} + 2r u_r + \exp(2\varepsilon\theta)(u_{\theta\theta} + \varepsilon u_\theta) = 0.$$

It is easy to check that the current density vector

$$\mathbf{P} \equiv \left(-r^2\exp(-\varepsilon\theta)u_r u_\theta, \tfrac{1}{2}r^2\exp(-\varepsilon\theta)u_r^2 - \tfrac{1}{2}\exp(\varepsilon\theta)u_\theta^2\right)$$

satisfies the μ-conservation law

$$D_i\mathbf{P}_i = -\varepsilon\mathbf{P}_2.$$

In agreement with theorem 6, in this case also the (standard) conservation law $D_i \widetilde{\mathbf{P}}^i = 0$ holds, with

$$\widetilde{\mathbf{P}} \equiv \left(-r^2 u_r u_\theta, \tfrac{1}{2} r^2 u_r^2 - \tfrac{1}{2} \exp(2\varepsilon\theta) u_\theta^2\right)$$

thanks to the (local, see [10]) equivalence to the standard symmetry vector field $\widetilde{X} = \gamma X = \exp(\varepsilon\theta)\partial/\partial\theta$.

Example 7. This is another modification of the Lagrangian of the Laplace equation. The following Lagrangian for the function $u = u(x, y)$

$$\mathcal{L} = \tfrac{1}{2} \exp(2u) \left(u_x^2 + u_y^2\right) + \tfrac{1}{3} \exp(3u) u_y^3$$

is clearly not invariant under $X = \partial/\partial u$, but it turns to be μ-invariant defining $\mu = -u_x \, \mathrm{d}x - u_y \, \mathrm{d}y$. The Euler–Lagrange equation is the PDE

$$u_{xx} + u_{yy} + u_x^2 + u_y^2 + 2\exp(u)u_y \left(u_y^2 + u_{yy}\right) = 0.$$

Rather than the expression of the μ-conserved current, it can be interesting in this case to remark that, according to the discussion in subsection 6.3, there is, as in the above example, an equivalent exact symmetry $\widetilde{X} = \exp(-u)X$ for this Lagrangian; as a consequence, the Euler–Lagrange equation turns out to be *both* exactly symmetric under $\widetilde{X}$ and μ-symmetric under X. Indeed, introducing the quantities $\alpha = u_x \exp(u)$, $\beta = u_y \exp(u)$, which are invariant under $\widetilde{X}$ (and μ-invariant under X), the Euler–Lagrange equation takes the invariant form

$$\alpha_x + \beta_y + 2\beta\beta_y = 0.$$

Clearly, now, all usual procedures for symmetric equations can be applied.

Example 8. The Lagrangian for the two variables $q_1(t)$, $q_2(t)$

$$\mathcal{L} = \tfrac{1}{2}\dot{q}_1^2 + \tfrac{1}{2}q_1^2(\dot{q}_2 - q_2)^2$$

is μ-invariant under

$$X = \frac{\partial}{\partial q_2} \qquad \text{with} \qquad \Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The μ-conservation law is

$$D_t \mathbf{P} = -\mathbf{P},$$

where $\mathbf{P} = q_1^2(\dot{q}_2 - q_2)$. Integrating this μ-conservation law one gets

$$\mathbf{P} = q_1^2(\dot{q}_2 - q_2) = c_1 \exp(-t) \quad (c_1 = \text{const})$$

and, from the Euler–Lagrange equations, one gets the other equation

$$q_1^3 \ddot{q}_1 = c_1^2 \exp(-2t).$$

It is now easy to obtain the solution

$$q_1(t) = \sqrt{2c_1} \exp(-t/2), \qquad q_2(t) = c_2 \exp(t) - \tfrac{1}{2}.$$

Example 9. The following Lagrangian for $u = u(x, y)$, $v(x, y)$

$$\mathcal{L} = \tfrac{1}{2} v^2 (u_x - u)^2 + \tfrac{1}{2}(u_y - u)^2 + v_x v_y$$

is μ-invariant under

$$X = \frac{\partial}{\partial u}$$

with $\Lambda_1 = \Lambda_2 = I$. The μ-conservation law is

$$D_x \mathbf{P}_1 + D_y \mathbf{P}_2 = -\mathbf{P}_1 - \mathbf{P}_2$$

where

$$\mathbf{P} = (v^2(u_x - u), u_y - u).$$

One can look for a possible solution imposing e.g. $\mathbf{P}_1 = 0$, and then $\mathbf{P}_2 = A(x) \exp(-y)$ where $A(x)$ is a function to be determined. It is now simple to verify that the Euler–Lagrange equations are solved by $u = \exp(x)(c_1 \exp(y) + c_2 \exp(-y))$ and v any solution to the equation $v_{xy} = 0$. Putting instead $\mathbf{P}_2 = 0$, then a solution is given e.g. by $u = -\exp(y), v = \exp(-x/2) \exp(-e^{2y}/2)$.

9. Final remarks

We have presented several aspects and different points of view concerning μ-symmetries and their relationships with μ-conservation laws and with standard symmetries as well.

The notion of 'conservation law' is a classical theme of theoretical and mathematical physics which certainly does not need to be emphasized here (see e.g. [17–20, 31, 35]). Several attempts are also present in the literature toward generalizations (or weakened forms) of this notion, and we would like to quote in particular [2] (see also [3–5, 8] and references therein) for physical applications, where attention is focused on nonlocal conservation laws.

The general relevance of the Lagrangian approach is also witnessed by the efforts made by various authors in order to embed in a variational context also generic (non-variational) problems (see e.g. [15, 16]).

It should be mentioned, in connection with [2], that a nontrivial relation between λ- and μ-symmetries on the one hand and nonlocal symmetries on the other has been determined [9]; this suggests there could be relations between our present results and those in [2].

Last but not least, we like to mention that the results given here have been reinterpreted by P Morando in terms of standard conservation laws under a deformed Lie derivative, coherently with her geometrical approach to μ-prolongations and symmetries [23].

Acknowledgments

References

[1] Amaldi U and Levi-Civita T 1924 *Lezioni di Meccanica Razionale* (Bologna: Zanichelli)
[2] Anco S C and Bluman G 1996 Derivation of conservation laws from nonlocal symmetries *J. Math. Phys.* **37** 2361–75
    Anco S C and Bluman G 1997 Nonlocal symmetries and nonlocal conservation laws of Maxwell equations *J. Math. Phys.* **38** 3508–32
[3] Anco S C 2003 Conservation laws of scaling invariant field equations *J. Phys. A: Math. Gen.* **36** 8623–38
[4] Anco S C and Bluman G 2002 Direct construction method for conservation laws of partial differential equations: I *Eur. J. Appl. Math.* **13** 545–66

Anco S C and Bluman G 2002 Direct construction method for conservation laws of partial differential equations: Part II *Eur. J. Appl. Math.* **13** 567–85

[5] Anco S C and The D 2005 Symmetries, conservation laws and cohomology of Maxwell equations using potentials *Acta Appl. Math.* **89** 1–52

[6] Arnold V I 1983 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)

[7] Birkhoff G D 1927 *Dynamical Systems* (New York: Chelsea)

[8] Bluman G, Temuerchaolu and Anco S C 2006 New conservation laws obtained directly from symmetry action on a known conservation law *J. Math. Anal. Appl.* **322** 233–50

[9] Catalano Ferraioli D 2007 Nonlocal aspects of λ-symmetries and ODE's reduction *J. Phys. A: Math. Theor.* **40** 5479–89

[10] Cicogna G, Gaeta G and Morando P 2004 On the relation between standard and μ-symmetries for PDEs *J. Phys. A: Math. Gen.* **37** 9467–86

[11] Gaeta G 1994 *Nonlinear Symmetries and Nonlinear Equations* (Dordrecht: Kluwer)

[12] Gaeta G 2005 Lambda and mu-symmetries *Symmetry and Perturbation Theory (SPT2004)* ed G Gaeta (Singapore: World Scientific)

[13] Gaeta G Frame invariance for differential equations and μ-prolongations *Int. J. Geom. Methods Mod. Phys.* at press

[14] Gaeta G and Morando P 2004 On the geometry of lambda-symmetries and PDEs reduction *J. Phys. A: Math. Gen.* **37** 6955–75

[15] Ibragimov N H and Kolsrud T 2004 Lagrangian approach to evolution equations: symmetries and conservation laws *Nonlinear Dyn.* **36** 29–40

[16] Ibragimov N H 2006 Integrating factors, adjoint equations and Lagrangians *J. Math. Anal. Appl.* **318** 742–57

[17] Kosmann-Schwarzbach Y (avec la collaboration de L Meersseman) 2004 *Les Théorèmes de Noether: invariance et lois de Conservation au XX^{e}me siècle* Les Editions de l'Ecole Polytechnique, Palaiseau

[18] Krasil'schik I S and Vinogradov A M (ed) 1999 *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics* (Providence, RI: American Mathematical Society)

[19] Landau L D and Lifshitz I M 1955 *Mechanics (Theoretical Physics vol 1)* (London: Pergamon)

[20] Landau L D and Lifshitz I M 1955 *Field Theory (Theoretical Physics vol 2)* (London: Pergamon)

[21] Levi Civita T 1899 Interpretazione gruppale degli integrali di un sistema canonico *R. Accad. Lincei. Ser.* III **8** 235–8

[22] Levi Civita T 1901 Sulla determinazione di soluzioni particolari di un sistema canonico quando se ne conosce qualche integrale o relazione invariante *R. Accad. Lincei Ser.* V **10** 3–9 and 35–41

[23] Morando P Deformation of differential operators and μ-symmetries *J. Phys. A: Math. Theor.* at press
Morando P Deformation of Lie derivatives and μ-symmetries *Symmetry and Perturbation Theory (SPT2007)* ed G Gaeta (Singapore: World Scientific) at press

[24] Muriel C and Romero J L *et al* 2001 New method of reduction for ordinary differential equations *IMA J. Appl. Math.* **66** 111–25
Muriel C and Romero J L 2001 C^∞-symmetries and nonsolvable symmetry algebras *IMA J. Appl. Math.* **66** 477–98

[25] Muriel C and Romero J L 2002 Prolongations of vector fields and the invariants-by-derivation property *Theor. Math. Phys.* **133** 1565–75
Muriel C and Romero J L 2003 C^∞-symmetries and reduction of equations without Lie point symmetries *J. Lie theory* **13** 167–88

[26] Muriel C and Romero J L 2002 C^∞ symmetries and integrability of ordinary differential equations *Proc. I Colloquium on Lie Theory and Applications* ed I Bajo and E Sanmartin (Vigo: Publicacións da Universidade de Vigo)

[27] Muriel C, Romero J L and Olver P J 2006 Variational C^∞ symmetries and Euler–Lagrange equations *J. Differ. Equ.* **222** 164–84

[28] Nakahara M 1990 *Geometry, Topology and Physics* (Bristol: Institute of Physics Publishing)

[29] Nash C and Sen S 1983 *Topology and Geometry for Physicists* (London: Academic)

[30] Noether E 1918 Invariante variationsprobleme *Nachr. Konig. Gesell. Wiss. Gottingen (Math.-Phys. Kl.)* pp 235–7 (A reliable French translation is included in [17])

[31] Olver P J 1986 *Application of Lie Groups to Differential Equations* (Berlin: Springer)

[32] Pucacco G 2005 On Birkhoff method for integrable Lagrangian systems *Symmetry and Perturbation Theory (SPT2004)* ed G Gaeta (Singapore: World Scientific)

[33] Pucacco G and Rosqvist K *et al* 2005 Configurational invariants of Hamiltonian systems *J. Math. Phys.* **46** 052902

[34] Pucci E and Saccomandi G 2002 On the reduction methods for ordinary differential equations *J. Phys. A: Math. Gen.* **35** 6145–55

[35] Sarlet W and Cantrijn F 1981 Generalizations of Noether's theorem in classical mechanics *SIAM Rev.* **23** 467–94

[36] Sarlet W, Leach P G L and Cantrijn F 1985 First integrals versus configurational invariants and a weak form of complete integrability *Physica* D **17** 87–98

[37] Stephani H 1989 *Differential Equations: Their Solution Using Symmetries* (Cambridge: Cambridge University Press)